The Negative Signature of Some Hermitian Matrices

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ABSTRACT

It is proved that the range of the negative signature of the hermitian completions of the partial matrix

$$\begin{bmatrix} A & B & \\ B^* & C & D \\ & D^* & E \end{bmatrix}$$

is an interval, and explicit formulae for the bounds of this interval are obtained. As an application the solvability of the problem of lifting with prescribed negative signature of defect is settled.

1. INTRODUCTION

In connection with problems in dilation theory and interpolation of meromorphic functions such as those formulated in [1] or [9], we considered in [3] a problem of lifting with minimal negative signature of defect, which in the simplest formulation reads as follows: assume that there are given two block matrices

$$T_r = [\,T \quad X\,], \qquad T_c = [\,T \quad Y\,]^t, \tag{1.1}$$

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where t stands for the matrix transpose, such that

$$\chi^-(I - T_r^*T_r) = \chi^-(I - T_c^*T_c) = \chi. \tag{1.2}$$

It is asked whether there exists a lifting of T_r and T_c,

$$\tilde{T} = \begin{bmatrix} T & X \\ Y & Z \end{bmatrix}, \tag{1.3}$$

with the property

$$\chi^-(I - \tilde{T}^*\tilde{T}) = \chi. \tag{1.4}$$

The answer (Theorem 1.1 in [3]) is the following: define two spaces

$$\mathscr{L}_1 = P_{\ker(I-T^*T)}\mathscr{R}(Y^*), \qquad \mathscr{L}_2 = P_{\ker(I-TT^*)}\mathscr{R}(X); \tag{1.5}$$

then there exists $\tilde{T}$ satisfying (1.3) and (1.4) if and only if

$$T\mathscr{L}_1 = \mathscr{L}_2. \tag{1.6}$$

Using the matrices T_r and T_c in (1.1), we can define a hermitian partial matrix

$$K = \begin{bmatrix} I & 0 & Y & \\ 0 & I & T & X \\ Y^* & T^* & I & 0 \\ & X^* & 0 & I \end{bmatrix}, \tag{1.7}$$

and by means of Frobenius-Schur factorizations we obtain

$$\chi^-\left(\begin{bmatrix} I & 0 & Y \\ 0 & I & T \\ Y^* & T^* & I \end{bmatrix}\right) = \chi^-(I - T_c^*T_c) \tag{1.8}$$

and

$$\chi^-\left(\begin{bmatrix} I & T & X \\ T^* & I & 0 \\ X^* & 0 & I \end{bmatrix}\right) = \chi^-(I - T_r^*T_r). \tag{1.9}$$

Moreover, if $\tilde{T}$ is the matrix in (1.3) and $K(Z)$ is the hermitian completion of K with Z, we have

$$\chi^{-}(K(Z)) = \chi^{-}(I - \tilde{T}^{*}\tilde{T}). \tag{1.10}$$

Thus, we are led to consider the following problem: assume that we are given a partial Hermitian matrix

$$K = \begin{bmatrix} A & B & \\ B^{*} & C & D \\ & D^{*} & E \end{bmatrix} \tag{1.11}$$

and a nonnegative integer χ. Does there exist a hermitian completion

$$K(Z) = \begin{bmatrix} A & B & Z \\ B^{*} & C & D \\ Z^{*} & D^{*} & E \end{bmatrix} \tag{1.12}$$

such that $\chi^{-}(K(Z)) = \chi$? This is the problem we are concerned with.

The calculation of the negative signature of hermitian completions of partial matrices was done in [6, 8, 5]. In all these papers the nonsingularity of all principal Hermitian submatrices appeared as a technical condition. For the case of one-step completions (i.e., A and E are scalars) the general case was considered in [4, 7].

The main result of our paper is Theorem 4.1, which shows that the range of $\chi^{-}(K(Z))$ is an interval $[\chi_{\min}, \chi_{\max}]$ in $\mathbb{Z}$, where $\chi_{\min}$ and $\chi_{\max}$ are explicitly computed in terms of the data of the problem (i.e. the matrices A, B, C, D, and E). This permits us to find necessary and sufficient conditions in order for there to exist hermitian completions with minimal signature (see Corollary 4.4). Thus, the phenomenon discovered in [3] turns out to be more general, namely, the analogue of the spaces $T\mathscr{L}_1$ and $\mathscr{L}_2$ in (1.5) appears as the pair of subspaces $\mathscr{R}(P_{\ker C} D)$ and $\mathscr{R}(P_{\ker C} B^{*})$, and the spatial positions of these spaces contain the basic information about the minimal negative signature of hermitian completions of K (see Corollaries 4.4–4.6).

The approach adopted in this paper requires a careful investigation of 2×2 block matrices. We do that in Section 3, where there is produced a formula for computing the negative signature of a 2×2 block matrix (see Proposition 3.1). This has much in common with the technique used in [11]. In Section 2 we fix some notation and recall some results, most of them from [3], which we need here.

In Section 5 we apply the results obtained for hermitian completions to the lifting problem with control of the negative signature of defect, following the pattern explained at the beginning. The results obtained in this section for the finite-dimensional case are true also for infinite-dimensional spaces, as will be shown elsewhere.

2. NOTATION AND SOME PRELIMINARY RESULTS

The objects appearing in this paper are matrices, but despite the usual understanding, we will regard them as linear operators acting in finite-dimensional Hilbert spaces. These matrices will be usually represented as block matrices with respect to specified decompositions of the domains and ranges.

Let $A \in \mathscr{L}(\mathscr{H})$ be a Hermitian matrix. Denote by $S_A = \operatorname{sgn} A$ the hermitian partial isometry appearing in the polar decomposition of A: $A = S_A|A|$, $\operatorname{Ker} S_A = \operatorname{Ker} A$. The *signatures* of A are defined as follows:

$$\chi^{\pm}(A) = \dim\ker(I \mp S_A), \qquad \chi^0(A) = \dim\ker A. \tag{2.1}$$

Recall that a *symmetry* J is a matrix such that $J^* = J = J^{-1}$. Then S_A is a symmetry when considered as acting on the space $\mathscr{R}(A)$ (the range of A).

The signature numbers $\chi^-(A)$, $\chi^+(A)$, and $\chi^0(A)$ can also be expressed as the dimensions of the spectral subspaces (or, equivalently, the numbers of the eigenvalues, counted according to their multiplicities) corresponding to the negative semiaxis, the positive semiaxis, and the null point, respectively.

In order to simplify the formulae we define also the following notation: for hermitian matrices $H_1 \in \mathscr{L}(\mathscr{H}_1)$ and $H_2 \in \mathscr{L}(\mathscr{H}_2)$ we write $H_1 = T(H_2, U)$ if $H_1 = U^*H_2U$, where $U \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ has the property $\mathscr{R}(U) \supseteq \mathscr{R}(H_2)$.

In this paper we will make intensive use of three results proved in [2] and [3] in the operatorial case.

LEMMA 2.1. *If* $H_1 = T(H_2, U)$ *then* $\chi^{\pm}(H_1) = \chi^{\pm}(H_2)$.

LEMMA 2.2. *Let* $A \in \mathscr{L}(\mathscr{H})$ *be a hermitian matrix and* $J \in \mathscr{L}(\mathscr{K})$ *be a symmetry. Then*

$$\begin{aligned}\{\chi^-(A - Z^*JZ) \mid Z \in \mathscr{L}(\mathscr{H}, \mathscr{K})\} &= \{\chi \in \mathbb{Z} \mid \max\{0, \chi^-(A) - \chi^-(J)\} \leqslant \chi \leqslant \chi^-(A) \\ &\qquad + \min\{\chi^+(J), \chi^+(A) + \chi^0(A)\}\}.\end{aligned}$$

The third result is based on the analysis of the matrix equation

$$B^*Z + Z^*B = C \tag{2.2}$$

which was done in [10].

LEMMA 2.3. *For given $B \in \mathscr{L}(\mathscr{H}, \mathscr{K})$ and hermitian $C \in \mathscr{L}(\mathscr{H})$, the equation* (2.2) *is solvable if and only if $P_{\ker B} C|\ker B = 0$; in particular,*

$$\begin{aligned}&\{\chi^{\pm}(C - B^*Z - Z^*B) | Z \in \mathscr{L}(\mathscr{H}, \mathscr{K})\}\\ &= \{\chi \in \mathbb{Z} | \chi^{\pm}(P_{\ker B} C|\ker B) \leqslant \chi\\ &\leqslant \chi^{\pm}(P_{\ker B} C|\ker B) + \operatorname{rank} B\}.\end{aligned}$$

For a subspace $\mathscr{G}$ of a space $\mathscr{H}$, we denote by $P_{\mathscr{G}}$ the orthogonal projection of $\mathscr{H}$ onto $\mathscr{G}$.

In this paper we will frequently use Frobenius-Schur factorizations, i.e., for matrices of appropriate dimensions A, B, C, D, if A is invertible, then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & 0 \\ CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I & A^{-1}B \\ 0 & I \end{bmatrix}, \tag{2.3}$$

and if D is invertible, then

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} I & 0 \\ D^{-1}C & I \end{bmatrix}. \tag{2.4}$$

For later use we also record a well-known fact.

LEMMA 2.4. *Let $S \in \mathscr{L}(\mathscr{H})$ be a hermitian matrix and $B \in \mathscr{L}(\mathscr{H}, \mathscr{K})$. Then the following factorization holds:*

$$\begin{bmatrix} 0 & B \\ B^* & S \end{bmatrix} = \begin{bmatrix} B & 0 \\ \frac{1}{2}S & I \end{bmatrix} \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} B^* & \frac{1}{2}S \\ 0 & I \end{bmatrix}; \tag{2.5}$$

in particular, if B is invertible, then $\begin{bmatrix} 0 & B \\ B^* & S \end{bmatrix}$ *is invertible and*

$$\begin{bmatrix} 0 & B \\ B^* & S \end{bmatrix}^{-1} = \begin{bmatrix} -B^{*-1}SB^{-1} & B^{*-1} \\ B^{-1} & 0 \end{bmatrix}. \tag{2.6}$$

3. THE NEGATIVE SIGNATURE OF 2×2 BLOCK MATRICES

Let $A \in \mathscr{L}(\mathscr{H})$ and $C \in \mathscr{L}(\mathscr{K})$ be hermitian matrices, and $B \in \mathscr{L}(\mathscr{K}, \mathscr{H})$. In this section we are interested in computing the negative signature of the hermitian matrix

$$H = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}. \tag{3.1}$$

In general, for a matrix X we denote by X^{-1} the generalized inverse, i.e. $X^{-1} \in \mathscr{L}(\mathscr{R}(X), \mathscr{R}(X^*))$.

PROPOSITION 3.1. *Let H be the hermitian matrix given by* (3.1), *and denote $B_1 = P_{\mathscr{R}(A)}B$, $B_2 = P_{\ker A}B$. Then*

$$\chi^-(H) = \chi^-(A) + \operatorname{rank} B_2 + \chi^-\big(P_{\ker B_2}(C - B_1^*A^{-1}B_1)|\ker B_2\big).$$

Proof. The matrix H has the representation

$$H = \begin{bmatrix} A & 0 & B_1 \\ 0 & 0 & B_2 \\ B_1^* & B_2^* & C \end{bmatrix}$$

with respect to the decomposition $\mathscr{R}(A) \oplus \ker A \oplus \mathscr{K}$. Decomposing again $\ker A = \mathscr{R}(B_2) \oplus \ker B_2^*$ and $\mathscr{K} = \mathscr{R}(B_2^*) \oplus \ker B_2$, we have

$$H = \begin{bmatrix} A & 0 & 0 & B_{11} & B_{12} \\ 0 & 0 & 0 & B_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ B_{11}^* & B_2^* & 0 & C_{11} & C_{12} \\ B_{12}^* & 0 & 0 & C_{12}^* & C_{22} \end{bmatrix}.$$

Now define

$$H' = \begin{bmatrix} S_A & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & P_{\mathscr{R}(B_2^*)} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & P_{\mathscr{R}(B_2^*)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \operatorname{sgn}\left(C_{22} - B_{12}^* A^{-1} B_{12}\right) \end{bmatrix}$$

and

$$V = \begin{bmatrix} |A|^{1/2} & 0 & 0 & S_A |A|^{1/2} B_{11} & S_a |A|^{1/2} B_{12} \\ 0 & B_2^* & 0 & \frac{1}{2}\left(C_{11} - B_{11}^* A^{-1} B_{11}\right) & C_{12} - B_{11}^* A^{-1} B_{12} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & |C_{22} - B_{12}^* A^{-1} B_{12}|^{1/2} \end{bmatrix},$$

and then a straightforward computation shows that $H = T(H', V)$. Applying Lemma 2.1, we obtain the required formula. ■

Proposition 3.1 makes it possible to obtain the negative signature for some completions of partial hermitian matrices.

COROLLARY 3.2. *Let $A \in \mathscr{L}(\mathscr{H})$ and $B \in \mathscr{L}(\mathscr{K}, \mathscr{H})$ be given matrices, $A = A^*$. Then*

$$\left\{ \chi^- \left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \right) \middle| C \in \mathscr{L}(\mathscr{K}), C = C^* \right\}$$

$$= \{ \chi \in \mathbb{Z} | \chi^-(A) + \operatorname{rank} B_2 \leqslant \chi \leqslant \chi^-(A) + \dim \mathscr{K} \}.$$

Proof. Direct consequence of Proposition 3.1. ■

COROLLARY 3.3. *Let $A \in \mathscr{L}(\mathscr{H})$ and $C \in \mathscr{L}(\mathscr{K})$ be given hermitian matrices. Then*

$$\left\{ \chi^- \left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \right) \middle| B \in \mathscr{L}(\mathscr{K}, \mathscr{H}) \right\}$$

$$= \{ \chi \in \mathbb{Z} | \max\{ \chi^-(A), \chi^-(C) \} \leqslant \chi \leqslant \chi^-(A) + \chi^-(C) + \min\{ \chi^0(A) + \chi^+(A), \chi^0(C) + \chi^+(C) \} \}.$$

Proof. Let $B \in \mathscr{L}(\mathscr{K},\mathscr{H})$ be a given matrix, and denote

$$\chi = \chi^-\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right).$$

Using Proposition 3.1 and Lemma 2.2, we immediately obtain

$$\chi \geqslant \max\{\chi^-(A), \chi^-(C)\}. \tag{3.2}$$

In order to prove the other inequality, notice first that

$$\operatorname{rank} B_2 \leqslant \min\{\chi^0(A), \dim \mathscr{K}\}, \tag{3.3}$$

and using Lemma 2.2 and Proposition 3.1, we have

$$\begin{aligned}
\chi \leqslant \chi^-(A) + \operatorname{rank} B_2 + \min\{\chi^-(C) + \min\{\chi^+(A), \chi^+(C) \\
+\chi^0(C)\}, \dim \mathscr{K} - \operatorname{rank} B_2\} \\
= \chi^-(A) + \chi^-(C) + \min\{\min\{\chi^+(A), \chi^+(C) + \chi^0(C)\} \\
+\operatorname{rank} B_2, \chi^+(C) + \chi^0(C)\} \\
\leqslant \chi^-(A) + \chi^-(C) + \min\{\chi^0(A) + \chi^+(A), \chi^0(C) + \chi^+(C)\}.
\end{aligned}$$

From (3.2) and the above inequality one of the required inclusions follows.

For proving the converse inclusion, it follows using Proposition 3.1 that we have to show that for any integer χ satisfying the inequalities

$$\max\{0, \chi^-(C) - \chi^-(A)\} \leqslant \chi \leqslant \chi^-(C) + \min\{\chi^0(A) + \chi^+(A), \chi^0(C) + \chi^+(C)\} \tag{3.4}$$

we can find $B_2 \in \mathscr{L}(\mathscr{K}, \ker A)$ and $B_{12} \in \mathscr{L}(\ker B_2, \mathscr{R}(A))$ such that

$$\chi = \operatorname{rank} B_2 + \chi^-\left(C_{22} - B_{12}^* A^{-1} B_{12}\right) \tag{3.5}$$

(recall that the notation is as in the proof of Proposition 3.1). Let $C = C_+ - C_-$ be the Jordan decomposition of C. As a general pattern, we will always choose B_2 such that

$$\mathscr{R}(B_2^*) = \mathscr{R}(B_2^*) \cap \ker C \oplus \mathscr{R}(B_2^*) \cap \mathscr{R}(C_+).$$

In particular, C has the representation

$$C = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{22} \end{bmatrix}$$

with respect to the decomposition $\mathscr{H} = \mathscr{R}(B_2^*) \oplus \ker B_2$. Denoting $\chi^0 = \dim[\mathscr{R}(B_2^*) \cap \ker C]$ and $\chi^+ = \dim[\mathscr{R}(B_2^*) \cap \mathscr{R}(C_+)]$, it follows that $\chi^-(C_{22}) = \chi^-(C)$, $\chi^0(C_{22}) = \chi^0(C) - \chi^0$, and $\chi^+(C_{22}) = \chi^+(C) - \chi^+$. Using these remarks and Lemma 2.2, our proof will be finished if we can find for any χ satisfying (3.4) two nonnegative integers $\chi^0 \leqslant \chi^0(C)$ and $\chi^+ \leqslant \chi^+(C)$ such that $\chi^0 + \chi^+ \leqslant \chi^0(A)$ and

$$\chi^0 + \chi^+ + \max\{0, \chi^-(C) - \chi^-(A)\} \leqslant \chi \leqslant \chi^-(C) + \min\{\chi^+(A) + \chi^+ + \chi^0, \chi^+(C) + \chi^0(C)\}. \tag{3.6}$$

When

$$\chi \leqslant \chi^-(C) + \min\{\chi^+(A), \chi^+(C) + \chi^0(C)\}, \tag{3.7}$$

then we can take $\chi^0 = \chi^+ = 0$. When (3.7) does not hold, we can represent

$$\chi = \chi^-(C) + \min\{\chi^+(A), \chi^+(C) + \chi^0(C)\} + k \tag{3.8}$$

with a nonnegative integer k such that

$$k \leqslant \min\{\chi^+(A) + \chi^0(A), \chi^+(C) + \chi^0(C)\} - \min\{\chi^+(A), \chi^+(C) + \chi^0(C)\}. \tag{3.9}$$

In this case, only when

$$\chi^+(A) + \chi^0(A) \geqslant \chi^+(C) + \chi^0(C) \geqslant \chi^+(A) \tag{3.10}$$

do, we need a more careful investigation.

Assuming that (3.10) holds, then (3.9) becomes

$$k \leqslant \chi^{+}(C) + \chi^{0}(C) - \chi^{+}(A). \tag{3.11}$$

Then, inserting (3.8) in (3.6), we reduce the problem to that of proving that the system with unknowns χ^0 and χ^+

$$\begin{aligned} &\chi^{0} + \chi^{+} \leqslant \chi^{-}(C) + \chi^{+}(A) + k - \max\{0, \chi^{-}(C) - \chi^{-}(A)\}, \\ &k \leqslant \chi^{0} + \chi^{+}, \\ &0 \leqslant \chi^{0} \leqslant \chi^{0}(C), 0 \leqslant \chi^{+} \leqslant \chi^{+}(C), \chi^{0} + \chi^{+} \leqslant \chi^{0}(A) \end{aligned} \tag{3.12}$$

always has solutions in $\mathbb{Z}$.

In order to do that, remark that

$$k \leqslant \chi^{-}(C) + \chi^{+}(A) + k - \max\{0, \chi^{-}(C) - \chi^{-}(A)\}, \tag{3.13}$$

and by (3.10) and (3.11), we obtain

$$0 \leqslant k \leqslant \min\{\chi^{0}(A), \chi^{0}(C) + \chi^{+}(C)\}. \tag{3.14}$$

In particular, (3.14) shows that the straight line whose equation is $\chi^0 + \chi^+ = k$ always intersects at least two of the sides of the rectangle ($0 \leqslant \chi^0 \leqslant \chi^0(C)$, $0 \leqslant \chi^+ \leqslant \chi^+(C)$) (see Figure 1). The coordinates of these intersection points are always integer and, by (3.13) and (3.8), they are solutions in $\mathbb{Z}$ of the system (3.12). ■

REMARK 3.4. It is obvious that

$$\begin{aligned} &\chi^{-}(A) + \chi^{-}(C) + \min\{\chi^{0}(A) + \chi^{+}(A), \chi^{0}(C) + \chi^{+}(C)\} \\ &\quad = \min\{\chi^{-}(A) + \dim\mathscr{K}, \chi^{-}(C) + \dim\mathscr{H}\}. \end{aligned}$$

For simplifying the notation in Corollary 3.3 and in the sequel, we define for two hermitian matrices $A \in \mathscr{L}(\mathscr{H})$ and $C \in \mathscr{L}(\mathscr{K})$ the numbers

$$m(A, C) = \max\{\chi^{-}(A), \chi^{-}(C)\} \tag{3.15}$$

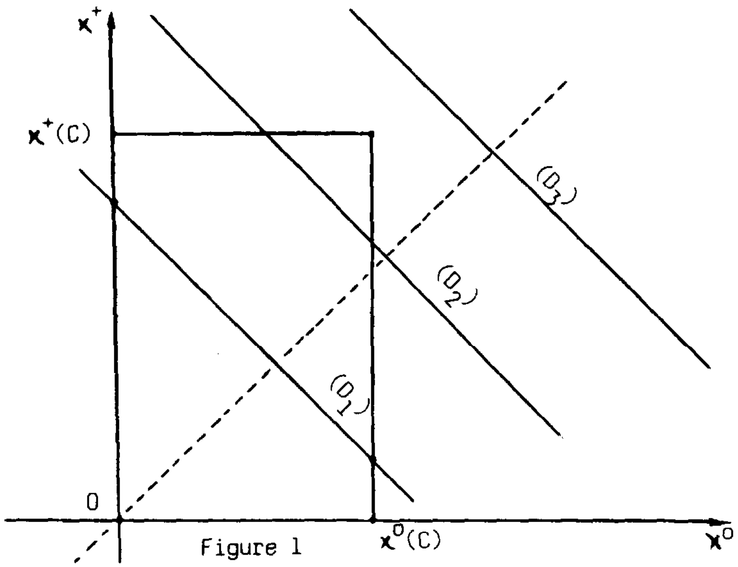

FIG. 1. (D_1): $\chi^0 + \chi^+ = \chi^0(A)$; (D_2): $\chi^0 + \chi^+ = \chi^-(C) + \chi^+(A) + k - \max\{0, \chi^-(C) - \chi^-(A)\}$; (D_3): $\chi^0 + \chi^+ = k$.

and

$$\begin{aligned} M(A,C) &= \chi^-(A) + \chi^-(C) + \min\{\chi^0(A) + \chi^+(A), \chi^0(C) + \chi^+(C)\} \\ &= \min\{\chi^-(A) + \dim\mathscr{K}, \chi^-(C) + \dim\mathscr{H}\}. \qquad (3.16) \end{aligned}$$

COROLLARY 3.5. *Let* $A \in \mathscr{L}(\mathscr{H})$, $A = A^*$, $C \in \mathscr{L}(\mathscr{K})$, $C = C^*$, *and* $B \in \mathscr{L}(\mathscr{K}, \mathscr{G})$ *be given. Then*

$$\left\{\chi^-\left(\begin{bmatrix} A & X \\ X^* & C - B^*Z - Z^*B \end{bmatrix}\right) \middle| X \in \mathscr{L}(\mathscr{K},\mathscr{H}), Z \in \mathscr{L}(\mathscr{K}, \mathscr{G})\right\}$$
$$= \{\chi \in \mathbb{Z} \mid m(A, P_{\ker B}C|\ker B) \leqslant \chi$$
$$\leqslant \operatorname{rank} B + M(A, P_{\ker B}C|\ker B)\}.$$

Proof. From Corollary 3.3 and Lemma 2.3 it follows that the analyzed set is an interval in $\mathbb{Z}$; hence it remains only to compute its bounds. But these also follow from Corollary 3.3 and Lemma 2.3. ■

REMARK 3.6. Concerning the negative signature of the hermitian matrix H in (3.1), there exists a formula which is dual to that obtained in Proposition 3.1, namely

$$\chi^{-}\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right) = \chi^{-}(C) + \operatorname{rank} B_2' + \chi^{-}\left(P_{\ker B_2'^*}(A - B_1'C^{-1}B_1'^*)|\ker B_2'^*\right),$$

where $B_2' = B|\ker C$, $B_1' = B|\mathscr{R}(C)$. From this and the formula in Proposition 3.1, we obtain

$$\begin{aligned}\chi^{-}(A) - \chi^{-}(C) = {} & \operatorname{rank} B_2' - \operatorname{rank} B_2 \\ & + \chi^{-}\left(P_{\ker B_2'^*}(A - B_1'C^{-1}B_1'^*)|\ker B_2'^*\right) \\ & - \chi^{-}\left(P_{\ker B_2}(C - B_1^*A^{-1}B_1)|\ker B_2\right).\end{aligned}$$

4. THE MAIN RESULT

Let us consider $A \in \mathscr{L}(\mathscr{H})$, $A = A^*$, $C \in \mathscr{L}(\mathscr{K})$, $C = C^*$, $E \in \mathscr{L}(\mathscr{G})$, $E = E^*$, and arbitrary $B \in \mathscr{L}(\mathscr{K},\mathscr{H})$ and $D \in \mathscr{L}(\mathscr{G},\mathscr{K})$. For any $Z \in \mathscr{L}(\mathscr{G},\mathscr{H})$ define a hermitian matrix

$$K(Z) = \begin{bmatrix} A & B & Z \\ B^* & C & D \\ Z^* & D^* & E \end{bmatrix}. \tag{4.1}$$

Our aim is to determine the range of $\chi^{-}(K(Z))$ when Z runs in $\mathscr{L}(\mathscr{G},\mathscr{H})$.

It is useful to introduce some more notation:

$$B_2 = B|\ker C, \qquad D_2 = P_{\ker B_2}D, \qquad D_1^2 = P_{\mathscr{R}(B_2^*)}D|\ker D_2, \tag{4.2}$$

$$A_{22} = P_{\ker B_2^*}\left(A - BP_{\mathscr{R}(C)}C^{-1}P_{\mathscr{R}(C)}B^*\right)|\ker B_2^*, \tag{4.3}$$

$$E_{22}^1 = P_{\ker D_1^2}\left(E - D^*P_{\mathscr{R}(C)}C^{-1}P_{\mathscr{R}(C)}D\right)|\ker D_1^2. \tag{4.4}$$

Also, recall the definitions of the functions m and M given in (3.15) and (3.16).

THEOREM 4.1. *With the notation stated in* (4.1)–(4.4), *we have*

$$\{\chi^{-}(K(Z)) \mid Z \in \mathscr{L}(\mathscr{G},\mathscr{H})\} = \{\chi \in \mathbb{Z} \mid \chi_{\min} \leqslant \chi \leqslant \chi_{\max}\},$$

where

$$\chi_{\min} = \chi^{-}(C) + \operatorname{rank} B_2 + \operatorname{rank} D_2 + m(A_{22}, E_{22}^{1})$$

and

$$\chi_{\max} = \chi^{-}(C) + \operatorname{rank} B_2 + \operatorname{rank} D_2 + \operatorname{rank} D_1^{2} + M(A_{22}, E_{22}^{1}).$$

Proof. Let us denote $B_1 = B|\mathscr{R}(C)$, $B_2 = B|\ker C$, $D_1 = P_{\mathscr{R}(C)}D$, and $D_2 = P_{\ker C}D$ [for the moment we forget about the definition of D_2 in (4.2)]. Then

$$K(Z) = \begin{bmatrix} A & B_1 & B_2 & Z \\ B_1^* & C & 0 & D_1 \\ B_2^* & 0 & 0 & D_2 \\ Z^* & D_1^* & D_2^* & E \end{bmatrix}.$$

Performing a Frobenius-Schur factorization and a reordering of rows and columns, we have

$$K(Z) = T\left(\begin{bmatrix} C & 0 & 0 & 0 \\ 0 & A - B_1C^{-1}B_1^* & B_2 & Z - B_1C^{-1}D_1 \\ 0 & B_2^* & 0 & D_2 \\ 0 & Z^* - D_1^*C^{-1}D_1 & D_2^* & E - D_1^*C^{-1}D_1 \end{bmatrix}, V\right),$$

where V is a certain invertible matrix. Taking Lemma 2.1 into account as well as the fact that the change of variable $Z \to Z - B_1C^{-1}D_1$ is bijective, it follows that, without restricting the generality, we can assume $C = 0$.

Assuming $C = 0$, we consider the decompositions

$$\mathscr{H} = \mathscr{R}(B) \oplus \ker B^*, \qquad \mathscr{K} = \mathscr{R}(B^*) \oplus \ker B, \qquad \mathscr{G} = \mathscr{R}(D_2^*) \oplus \ker D_2$$

and also

$$\ker B = \mathscr{R}(D_2) \oplus \ker D_2^*.$$

Hence we have the representation

$$K(Z) = \begin{bmatrix} A_1 & A_{12} & B & 0 & 0 & Z_1^1 & Z_1^2 \\ A_{12}^* & A_{22} & 0 & 0 & 0 & Z_2^1 & Z_2^2 \\ B^* & 0 & 0 & 0 & 0 & D_1^1 & D_1^2 \\ 0 & 0 & 0 & 0 & 0 & D_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ Z_1^{1*} & Z_2^{1*} & D_1^{1*} & D_2^* & 0 & E_{11} & E_{12} \\ Z_1^{2*} & Z_2^{2*} & D_1^{2*} & 0 & 0 & E_{12}^* & E_{22} \end{bmatrix}. \tag{4.5}$$

Let us remark that now we are in tune with the notation in (4.2)–(4.4) (of course, in our special case $C = 0$).

Further, by eliminating the rows and columns in (4.5) which are null and reordering rows and columns, we obtain a matrix

$$K'(Z) = \left[\begin{array}{cccc|cc} A_{11} & Z_1^1 & B & 0 & A_{12} & Z_1^2 \\ Z_1^{1*} & E_{11} & D_1^{1*} & D_2^* & Z_2^{1*} & E_{12} \\ B^* & D_1^1 & 0 & 0 & 0 & D_1^2 \\ 0 & D_2 & 0 & 0 & 0 & 0 \\ \hline A_{12}^* & Z_2^1 & 0 & 0 & A_{22} & Z_2^2 \\ Z_1^{2*} & E_{12}^* & D_1^{2*} & 0 & Z_2^{2*} & E_{22} \end{array}\right] \tag{4.6}$$

with $\chi^-(K'(Z)) = \chi^-(K(Z))$. Since the matrix

$$\begin{bmatrix} B & 0 \\ D_1^{1*} & D_2^* \end{bmatrix}$$

is invertible, it follows from Lemma 2.4 that the matrix

$$H(Z_1^1) = \begin{bmatrix} A_{11} & Z_1^1 & B & 0 \\ Z_1^{1*} & E_{11} & D_1^{1*} & D_2^* \\ B^* & D_1^1 & 0 & 0 \\ 0 & D_2 & 0 & 0 \end{bmatrix} \tag{4.7}$$

is invertible. Using Lemma 2.4 and Lemma 2.1, we have

$$\chi^-\left(H(Z_1^1)\right) = \operatorname{rank} B + \operatorname{rank} D_2. \tag{4.8}$$

Performing a Frobenius-Schur factorization in (4.6) with respect to the invertible matrix $H(Z_1^1)$, we obtain

$$K'(Z) = T\left(\begin{bmatrix} H(Z_1^1) & 0 \\ 0 & G(Z_1^2, Z_2^2) \end{bmatrix}; U\right),$$

where

$$G(Z_1^2, Z_2^2)$$

$$= \begin{bmatrix} A_{22} & Z_2^2 - A_{12}^* D_1^{2*} \\ Z_2^{2*} - D_1^2 A_{12} & E_{22} - \frac{1}{2} D_1^{2*}\left(A_{11} B^{*-1} D_1^2 + Z_1^2\right) \\ & -\frac{1}{2}\left(A_{11} B^{*-1} D_1^2 + Z_1^2\right)^* D_1^2 \end{bmatrix}.$$

In $G(Z_1^2, Z_2^2)$ we make the bijective changes of variables $Z_2^2 \to Z_2^2 + A_{12}^* D_1^{2*}$ and $Z_1^2 \to \frac{1}{2}(Z_1^2 - A_{11} B^{*-1} D_1^2)$. For finishing the proof we have only to use Lemma 2.1, Corollary 3.5, and (4.8). ■

Let us consider the partial hermitian matrix

$$K = \begin{bmatrix} A & B & \\ B^* & C & D \\ & D^* & E \end{bmatrix}. \tag{4.9}$$

By definition, the negative signature of this K is

$$\chi^-(K) = \max\left\{\chi^-\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right), \quad \chi^-\left(\begin{bmatrix} C & D \\ D^* & E \end{bmatrix}\right)\right\}. \tag{4.10}$$

Note now an alternative form of Theorem 4.1.

COROLLARY 4.2. *The numbers* $\chi_{\min}$ *and* $\chi_{\max}$ *in Theorem* 4.1 *can be expressed in terms of* $\chi^-(K)$ *by*

$$\chi_{\min} = \chi^-(K) + \min\left\{\operatorname{rank} D_2 + \max\left\{0, \chi^-\left(E_{22}^1\right) - \chi^-(A_{22})\right\},\right.$$
$$\left.\operatorname{rank} B_2 - \operatorname{rank} D_1^2 + \max\left\{0, \chi^-(A_{22}) - \chi^-\left(E_{22}^1\right)\right\}\right\}$$

and

$$\chi_{\max} = \chi^-(K) + \min\left\{\chi^0(A_{22}) + \chi^+(A_{22}), \chi^0\left(E_{22}^1\right) + \chi^+\left(E_{22}^1\right)\right\}$$
$$+ \min\left\{\operatorname{rank} D_2 + \operatorname{rank} D_1^2 + \chi^-\left(E_{22}^1\right), \operatorname{rank} B_2 + \chi^-(A_{22})\right\}.$$

Proof. By Proposition 3.1 we have

$$\chi^-\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right) = \chi^-(C) + \operatorname{rank} B_2 + \chi^-(A_{22}). \tag{4.11}$$

We claim that the following identity also holds:

$$\chi^-\left(\begin{bmatrix} C & D \\ D^* & E \end{bmatrix}\right) = \chi^-(C) + \operatorname{rank} D_2 + \operatorname{rank} D_1^2 + \chi^-\left(E_{22}^1\right). \tag{4.12}$$

In order to prove this claim we first notice that, as in the proof of Theorem 4.1, we can assume without loss of generality that $C = 0$. Then, with the notation as in (4.5), we have

$$
\begin{aligned}
\chi^-\left(\begin{bmatrix} 0 & D \\ D^* & E \end{bmatrix}\right) &= \chi^-\left(\begin{bmatrix} 0 & 0 & 0 & D_1^1 & D_1^2 \\ 0 & 0 & 0 & D_2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ D_1^{1*} & 0 & 0 & E_{11} & E_{12} \\ D_1^{2*} & D_2^* & 0 & E_{12}^* & E_{22} \end{bmatrix}\right) \\
&= \chi^-\left(\begin{bmatrix} E_{11} & D_2^* & E_{12} & D_1^{1*} \\ D_2 & 0 & 0 & 0 \\ E_{12}^* & 0 & 0 & D_1^2 \\ D_1^1 & 0 & D_1^{2*} & E_{22} \end{bmatrix}\right) \\
&= \chi^-\left(\begin{bmatrix} E_{11} & D_2^* \\ D_2 & 0 \end{bmatrix}\right) + \chi^-\left(\begin{bmatrix} 0 & D_1^2 \\ D_1^{2*} & E_{22} \end{bmatrix}\right) \\
&= \operatorname{rank} D_2 + \operatorname{rank} D_1^2 + \chi^-\left(E_{22}^1\right),
\end{aligned}
$$

where, having always in mind Lemma 2.1, for the second equality we have carried out the cancellation of rows and columns which are null and a reordering of the remaining ones, for the third equality we had to perform a Frobenius-Schur factorization with respect to the invertible matrix

$$
\begin{bmatrix} E_{11} & D_2^* \\ D_2 & 0 \end{bmatrix},
$$

and finally, we have used Lemma 2.4 twice. Thus, we have proved our claim. From (4.11) and (4.12), using the formulae for $\chi_{\min}$ and $\chi_{\max}$ in Theorem 4.1, it is easy to finish the proof. ■

REMARK 4.3. From the definition of D_1^2 it follows that $\ker D_1^2 = \ker(P_{\ker C} D)$; in particular, E_{22}^1 is actually defined by a similar formula with respect to A_{22}. Also, the formulae for $\chi_{\min}$ and $\chi_{\max}$ obtained in Theorem 4.1 can be modified in such a way that the matrices

$$\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} C & D \\ D^* & E \end{bmatrix}$$

have symmetric roles.

Indeed, let us denote by P the orthogonal projection of $\ker C$ onto the range of $P_{\ker C} B^* = B_2^*$ and also denote by Q the orthogonal projection of $\ker C$ onto the range of $P_{\ker C} D$. Then notice that

$$\operatorname{rank} D_2 = \operatorname{rank} Q(I - P)$$

and

$$\operatorname{rank} D_1^2 = \operatorname{rank} P \wedge Q,$$

where $P \wedge Q$ denotes the orthogonal projection onto $\mathscr{R}(P) \cap \mathscr{R}(Q)$. Taking into account the decompositions

$$\mathscr{R}(P) = \mathscr{R}(P(I - Q)) \oplus \mathscr{R}(P \wedge Q)$$

and

$$\mathscr{R}(Q) = \mathscr{R}(Q(I - P)) \oplus \mathscr{R}(P \wedge Q),$$

we obtain

$$\operatorname{rank} B_2 = \operatorname{rank} P(I - Q) + \operatorname{rank} P \wedge Q$$

and

$$\operatorname{rank} D_2 + \operatorname{rank} D_1^2 = \operatorname{rank} Q.$$

Using all these, the formulae obtained in Theorem 4.1 become

$$\chi_{\min} = \chi^-(C) + \operatorname{rank} P(I - Q) + \operatorname{rank} Q(I - P)$$
$$+ \operatorname{rank} P \wedge Q + m\big(A_{22}, E_{22}^1\big)$$

and

$$\chi_{\max} = \chi^{-}(C) + \operatorname{rank} P + \operatorname{rank} Q + M\left(A_{22}, E_{22}^{1}\right).$$

We can now consider the problem whether there exists any completion of the partial matrix K given by (4.9) which preserves the negative signature, i.e. $\chi_{\min} = \chi^{-}(K)$.

COROLLARY 4.4. *In order for there to exist* $Z \in \mathscr{L}(\mathscr{G},\mathscr{H})$ *such that* $\chi^{-}(K(Z)) = \chi^{-}(K)$, *it is necessary and sufficient that at least one of the following alternatives holds*:

(i) *if* $\chi^{-}(A_{22}) \geqslant \chi^{-}(E_{22}^{1})$ *then* $\mathscr{R}(P_{\ker C} D) \subseteq \mathscr{R}(P_{\ker C} B^*)$;
(ii) *if* $\chi^{-}(A_{22}) \leqslant \chi^{-}(E_{22}^{1})$ *then* $\mathscr{R}(P_{\ker} CB^*) \subseteq \mathscr{R}(P_{\ker C} D)$.

Proof. Let us remark first that, just from the definition of D_1^2, it follows that $\operatorname{rank} D_1^2 \leqslant \operatorname{rank} B_2$. Now, from Corollary 4.2 it follows that $\chi_{\min} = \chi^{-}(K)$ if and only if at least one of the following conditions holds: either $D_2 = 0$ and $\chi^{-1}(E_{22}^{1}) \leqslant \chi^{-}(A_{22})$, or $\operatorname{rank} B_2 = \operatorname{rank} D_1^2$ and $\chi^{-}(E_{22}^{1}) \geqslant \chi^{-}(A_{22})$. It remains to notice that $D_2 = 0$ if and only if $\mathscr{R}(P_{\ker C} D) \subseteq \mathscr{R}(P_{\ker C} B^*)$, and also that $\operatorname{rank} B_2 = \operatorname{rank} D_1^2$ if and only if $\mathscr{R}(P_{\ker C} B^*) \subseteq \mathscr{R}(P_{\ker C} D)$. ∎

In some applications, the following condition appears:

$$\chi^{-}\left(\begin{bmatrix} A & B \\ B^* & C \end{bmatrix}\right) = \chi^{-}\left(\begin{bmatrix} C & D \\ D^* & E \end{bmatrix}\right) \left[= \chi^{-}(K)\right]. \tag{4.13}$$

COROLLARY 4.5. *If* (4.13) *holds, then*

$$\chi_{\min} = \chi^{-}(K) + \max\left\{\operatorname{rank} D_2, \operatorname{rank} B_2 - \operatorname{rank} D_1^2\right\}$$

and

$$\chi_{\max} = 2\chi^{-}(K) - \chi^{-}(C)$$
$$+ \min\left\{\chi^{0}(A_{22}) + \chi^{+}(A_{22}), \chi^{0}\left(E_{22}^{1}\right) + \chi^{+}\left(E_{22}^{1}\right)\right\}.$$

Proof. Taking into account (4.11) and (4.12), it follows that if (4.13) holds, then

$$\operatorname{rank} B_2 + \chi^-(A_{22}) = \operatorname{rank} D_2 + \operatorname{rank} D_1^2 + \chi^-\left(E_{22}^1\right).$$

It remains only to use Corollary 4.2. ■

COROLLARY 4.6. *If* (4.14) *holds, then there exists* Z *such that* $\chi^-(K(Z)) = \chi^-(K)$ *if and only if* $\mathscr{R}(P_{\ker C} D) = \mathscr{R}(P_{\ker C} B^*)$.

Proof. Indeed, by Corollary 4.5 it follows that $\chi_{\min} = \chi^-(K)$ if and only if $D_2 = 0$ and $\operatorname{rank} B_2 = \operatorname{rank} D_1^2$. But we have already seen that these are equivalent with $\mathscr{R}(P_{\ker C} D) = \mathscr{R}(P_{\ker C} B^*)$. ■

5. LIFTING WITH PRESCRIBED NEGATIVE SIGNATURE OF DEFECT

As we mentioned in the Introduction, Theorem 4.1 has several applications. Here we present one of them, to the problem of lifting with prescribed signature of defect.

Let $\mathscr{H}_1, \mathscr{H}_1', \mathscr{H}_2, \mathscr{H}_2'$ be Hilbert spaces of finite dimensions, and denote $\tilde{\mathscr{H}}_1 = \mathscr{H}_1 \oplus \mathscr{H}_1'$ and $\tilde{\mathscr{H}}_2 = \mathscr{H}_2 \oplus \mathscr{H}_2'$. Assume that there are given two matrices $T_r \in \mathscr{L}(\tilde{\mathscr{H}}_1, \mathscr{H}_2)$ and $T_c \in \mathscr{L}(\mathscr{H}_1, \tilde{\mathscr{H}}_2)$ such that

$$T_r | \mathscr{H}_1 = P_{\mathscr{H}_2} T_c = T \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2). \tag{5.1}$$

For a fixed nonnegative integer χ, the problem we are interested in is the following:

(P_χ) Determine, if any, matrices $\tilde{T} \in \mathscr{L}(\tilde{\mathscr{H}}_1, \tilde{\mathscr{H}}_2)$ such that $\tilde{T} | \mathscr{H}_1 = T_c$, $P_{\mathscr{H}_2} \tilde{T} = T_r$, and $\chi^-(I - \tilde{T}^* \tilde{T}) = \chi$.

We need now to recall some definitions. Conventionally, with $T \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ one associates the *defect matrices*

$$D_T = |I - T^*T|^{1/2}, \qquad D_{T^*} = |I - TT^*|^{1/2}, \tag{5.2}$$

the *sign matrices* of the defect,

$$J_T = \operatorname{sgn}(I - T^*T), \qquad J_{T^*} = \operatorname{sgn}(I - TT^*), \tag{5.3}$$

and the *defect spaces* $\mathscr{D}_T = \mathscr{R}(D_T)$ and $\mathscr{D}_{T^*} = \mathscr{R}(D_{T^*})$. Conventionally, J_T and J_{T^*} are viewed as acting on $\mathscr{D}_T$ and on $\mathscr{D}_{T^*}$, respectively.

From (5.1) we must have

$$T_r = [T \quad X], \qquad T_c = [T \quad Y]^t, \tag{5.4}$$

where $X \in \mathscr{L}(\mathscr{H}_1', \mathscr{H}_2)$ and $Y \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2')$ are uniquely determined. Moreover, there exist uniquely determined $\Lambda_1 \in \mathscr{L}(\mathscr{H}_1', \ \ker D_{T^*})$, $\Gamma_1 \in (\ker \Lambda_1, \mathscr{D}_{T^*})$ and $\Delta_1 \in \mathscr{L}(\mathscr{R}(\Lambda_1^*), \mathscr{D}_{T^*})$ such that

$$X = \begin{bmatrix} D_{T^*}\Gamma_1 & D_{T^*}\Delta_1 \\ 0 & \Lambda_1 \end{bmatrix}, \tag{5.5}$$

and similarly, there exist uniquely determined $\Lambda_2 \in \mathscr{L}(\ker D_T, \mathscr{H}_2')$, $\Gamma_2 \in \mathscr{L}(\mathscr{D}_T, \ker \Lambda_2^*)$, and $\Delta_2 \in \mathscr{L}(\mathscr{D}_T, \mathscr{R}(\Lambda_2))$ such that

$$Y = \begin{bmatrix} \Gamma_2 D_T & 0 \\ \Delta_2 D_T & \Lambda_2 \end{bmatrix}. \tag{5.6}$$

All these are objects appeared during the approach followed in [3]. Recall also the definitions of the functions m and M in (3.15) and (3.16).

THEOREM 5.1. *Problem* (P_χ) *has solutions if and only if* $\chi'_{\min} \leqslant \chi \leqslant \chi'_{\max}$, *where*

$$\chi'_{\min} = \chi^-(I - T^*T) + \operatorname{rank}\Lambda_2 + \operatorname{rank}(P_{\ker\Lambda_2}T^*\Lambda_1)$$

$$+ m(I - \Gamma_2 J_T \Gamma_2^*, I - \Gamma_1^* J_{T^*}\Gamma_1)$$

and

$$\chi'_{\max} = \chi^-(I - T^*T) + \operatorname{rank}\Lambda_2 + \operatorname{rank}(P_{\ker\Lambda_2}T^*\Lambda_1)$$

$$+ \dim[T\mathscr{R}(\Lambda_2^*) \cap \mathscr{R}(\Lambda_1)] + M(I - \Gamma_2 J_T \Gamma_2^*, I - \Gamma_1^* J_{T^*}\Gamma_1).$$

Proof. A matrix $\tilde{T} \in \mathcal{L}(\tilde{\mathcal{H}}_1, \tilde{\mathcal{H}}_2)$ such that $\tilde{T}|\mathcal{H}_c = T_c$ and $P_{\mathcal{H}_2}\tilde{T} = T_r$ must be of the form

$$\tilde{T} = \begin{bmatrix} T & X \\ Y & Z \end{bmatrix} \tag{5.7}$$

with respect to the decompositions $\tilde{\mathcal{H}}_1 = \mathcal{H}_1 \oplus \mathcal{H}_1'$ and $\tilde{\mathcal{H}}_2 = \mathcal{H}_2 \oplus \mathcal{H}_2'$, where only $Z \in \mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')$ has to be determined.

We consider the hermitian matrix.

$$A(Z) = \begin{bmatrix} I & 0 & Y & Z \\ 0 & I & T & X \\ Y^* & T^* & I & 0 \\ Z^* & X^* & 0 & I \end{bmatrix}, \tag{5.8}$$

which is represented as a block matrix with respect to the space $\mathcal{H}_2' \oplus \mathcal{H}_2 \oplus \mathcal{H}_1 \oplus \mathcal{H}_1'$. Performing a Frobenius-Schur factorization with respect to the identity matrix of the space $\tilde{\mathcal{H}}_1$ and using Lemma 2.1, it follows that

$$\chi^-(A(Z)) = \chi^-(I - \tilde{T}^*\tilde{T}); \tag{5.9}$$

hence, problem (P_χ) is solvable if and only if χ belongs to the range of $\chi^-(A(Z))$ when Z runs over $\mathcal{L}(\mathcal{H}_1', \mathcal{H}_2')$. Performing a Frobenius-Schur factorization of $A(Z)$ with respect to the identity on the space $\mathcal{H}_1$, we obtain the hermitian matrix

$$K(Z) = \begin{bmatrix} I & 0 & Y & Z \\ 0 & I & 0 & X \\ Y^* & 0 & I - T^*T & -T^*X \\ Z^* & X^* & -X^*T & I \end{bmatrix}, \tag{5.10}$$

and by Lemma 2.1 we always have $\chi^-(K(Z)) = \chi^-(A(Z))$. From what we have proved until now and using Theorem 4.1, it follows that problem (P_χ) has solution if and only if $\chi'_{\min} \leqslant \chi \leqslant \chi'_{\max}$, where $\chi'_{\min}$ and $\chi'_{\max}$ are those in Theorem 4.1 for $K(Z)$ given by (5.10).

It remains to compute the matrices defined in (4.2)–(4.4) in terms of the data of problem (P_χ). Now, using the representations (5.5) and (5.6) of X and Y, respectively, the objects defined in (4.2)–(4.4), in our case, are

$$B_2 = \Lambda_2, \qquad D_2 = -P_{\ker \Lambda_2} T^* \Lambda_1, \tag{5.11}$$

$$A_{22} = I - \Gamma_2 J_T \Gamma_2^*, \qquad E_{22}^1 = I - \Gamma_1^* J_{T^*} \Gamma_1. \tag{5.12}$$

Then, from the second equality in (5.11), we have

$$\operatorname{Ker} D_2 = \{x \in \mathscr{H}_1' \,|\, T^*\Lambda_1 x \in \mathscr{R}(\Lambda_2^*)\} = \{x \in \mathscr{H}_1' \,|\, \Lambda_1 x \in T\mathscr{R}(\Lambda_2^*)\}$$
$$= \ker \Lambda_1 \oplus \Lambda_1^{-1}[T\mathscr{R}(\Lambda_2^*) \cap \mathscr{R}(\Lambda_1)]$$

because T is unitary as acting between $\ker D_T$ and $\ker D_{T^*}$. This implies

$$D_1^2 = -P_{\mathscr{R}(\Lambda_2^*)}T^*\Lambda_1/\Lambda_1^{-1}[T\mathscr{R}(\Lambda_2^*) \cap \mathscr{R}(\Lambda_1)], \tag{5.13}$$

and then,

$$\operatorname{rank} D_1^2 = \dim[T\mathscr{R}(\Lambda_2^*) \cap \mathscr{R}(\Lambda_1)]. \tag{5.14}$$

Inserting (5.11), (5.12), and (5.14) in Theorem 4.1, we obtain the required formulae for $\chi'_{\min}$ and $\chi'_{\max}$. ∎

REMARK 5.2. Using Frobenius-Schur factorizations and Lemma 2.1, we obtain

$$\chi^-\left(\begin{bmatrix} I & 0 & Y \\ 0 & I & T \\ Y^* & T^* & I \end{bmatrix}\right) = \chi^-(I - T_c^*T_c) \tag{5.15}$$

and

$$\chi^-\left(\begin{bmatrix} I & T & X \\ T^* & I & 0 \\ X^* & 0 & I \end{bmatrix}\right) = \chi^-(I - T_r^*T_r). \tag{5.16}$$

From this and Proposition 3.1 we obtain

$$\chi^-(I - T_r^*T_r) = \chi^-(I - T^*T) + \chi^-(I - \Gamma_1^*J_{T^*}\Gamma_1) + \operatorname{rank} \Lambda_1 \tag{5.17}$$

and

$$\chi^-(I - T_c^*T_c) = \chi^-(I - T^*T) + \chi^-(I - \Gamma_2 J_T \Gamma_2^*) + \operatorname{rank} \Lambda_2. \tag{5.18}$$

These formulae were obtained in [3] by a different method.

REMARK 5.3. Using (5.15) and (5.16) and following the proof of Theorem 5.1, one can express $\chi'_{\min}$ and $\chi'_{\max}$ in terms of the number $\max\{\chi^-(I - T_r^*T_r), \chi^-(I - T_c^*T_c)\}$, similarly to Corollary 4.2.

An important case for problem (P_χ) is when $\chi = \chi^-$, where

$$\chi^- = \max\{\chi^-(I - T_r^*T_r), \chi^-(I - T_c^*T_c)\}, \tag{5.19}$$

the minimal negative signature of defect that one can expect to be preserved by the lifting.

COROLLARY 5.4. *Problem* (P_{χ^-}) *has solutions for* χ^- *in* (5.19) *if and only if one of the following alternatives holds*:

(i) *if* $\chi^-(I - \Gamma_2 J_T \Gamma_2^*) \geqslant \chi^-(I - \Gamma_1^* J_{T^*}\Gamma_1)$ *then* $T^*\mathscr{R}(\Lambda_1) \subseteq \mathscr{R}(\Lambda_2^*)$;
(ii) *if* $\chi^-(I - \Gamma_2 J_T \Gamma_2^*) \leqslant \chi^-(I - \Gamma_1^* J_{T^*}\Gamma_1)$ *then* $\mathscr{R}(\Lambda_2^*) \subseteq T^*\mathscr{R}(\Lambda_1)$.

Proof. As in the proof of Theorem 5.1, using Corollary 4.4. ■

In [3] we considered the special case when

$$\chi^-(I - T_r^*T_r) = \chi^-(I - T_c^*T_c) = \chi^-. \tag{5.20}$$

COROLLARY 5.5. *Assume that the condition* (5.20) *holds. Then problem* (P_χ) *has solutions if and only if* $\chi'_{\min} \leqslant \chi \leqslant \chi'_{\max}$, *where*

$$\chi'_{\min} = \chi^- + \max\left\{\operatorname{rank}\left(P_{\ker \Lambda_2} T^*\Lambda_1\right), \operatorname{rank}\left(P_{\ker \Lambda_1^*} T\Lambda_2^*\right)\right\}$$

and

$$\begin{aligned}\chi'_{\max} = 2\chi^- - \chi^-(I - T^*T) \\ + \min\{\chi^0(I - \Gamma_2 J_T \Gamma_2^*) + \chi^+(I - \Gamma_2 J_T \Gamma_2^*), \chi^0(I - \Gamma_1^* J_{T^*}\Gamma_1) \\ + \chi^+(I - \Gamma_1^* J_{T^*}\Gamma_1)\}.\end{aligned}$$

Proof. Apply Corollary 4.5, taking into account that $\operatorname{rank}\Lambda_2 - \dim[T\mathscr{R}(\Lambda_2^*) \cap \mathscr{R}(\Lambda_1)] = \operatorname{rank}(P_{\ker \Lambda_1^*} T\Lambda_2^*)$. ■

We can reobtain now (in the finite-dimensional case) the result from [3, Theorem 1.1].

COROLLARY 5.6. *Assume that* (5.20) *holds, and take* $\chi = \chi^-$. *Then problem* (P_χ) *has solutions if and only if* $T^*\mathscr{R}(\Lambda_1) = \mathscr{R}(\Lambda_2^*)$.

REMARK 5.7. Problem (P_χ) can be formulated in much more generality. On each of the Hilbert spaces $\mathscr{H}_1, \mathscr{H}_1', \mathscr{H}_2$, and $\mathscr{H}_2'$ there are fixed symmetries J_1, J_1', J_2, and J_2', respectively. Define $\tilde{J}_1 = J_1 \oplus J_1'$ and $\tilde{J}_2 = J_2 \oplus J_2'$, which are symmetries on $\tilde{\mathscr{H}}_1 = \mathscr{H}_1 \oplus \mathscr{H}_1'$ and $\tilde{\mathscr{H}}_2 = \mathscr{H}_2 \oplus \mathscr{H}_2'$, respectively. Considering T_r and T_c satisfying (5.1) and nonnegative integers χ_1 and χ_2, we formulate the problem:

$(\mathrm{P}_{\chi_1,\chi_2})$ Determine, if any, $\tilde{T} \in \mathscr{L}(\tilde{\mathscr{H}}_1, \tilde{\mathscr{H}}_2)$ such that $\tilde{T}|\mathscr{H}_1 = T_c$, $P_{\mathscr{H}_2}\tilde{T} = T_r$, $\chi^-(\tilde{J}_1 - \tilde{T}^*\tilde{J}_2\tilde{T}) = \chi_1$, and $\chi^-(\tilde{J}_2 - \tilde{T}\tilde{J}_1\tilde{T}^*) = \chi_2$.

Taking the identity

$$\chi^-(J_1 - T^*J_2T) + \chi^-(J_2) = \chi^-(J_2 - TJ_1T^*) + \chi^-(J_1) \quad (5.21)$$

into account, it follows that, in order to have a solution of problem $(\mathrm{P}_{\chi_1,\chi_2})$, it is necessary that

$$\chi_2 + \chi^-(J_1 - T^*J_2T) + \chi^-(J_1') = \chi_1 + \chi^-(J_2 - TJ_1T^*) + \chi^-(J_2'). \quad (5.22)$$

Then, with the notation from $(\mathscr{H}'$, consider the hermitian matrix

$$A(Z) = \begin{bmatrix} J_2' & 0 & Y & Z \\ 0 & J_2 & T & X \\ Y^* & T^* & J_1 & 0 \\ Z^* & X^* & 0 & J_1' \end{bmatrix}. \quad (5.23)$$

Reasoning as in the proof of Theorem 5.1, one obtains explicit formulae for the intervals where χ_1 and χ_2 must live in order that problem $(\mathrm{P}_{\chi_1,\chi_2})$ be solvable. All the details will be presented in a forthcoming paper treating the infinite-dimensional variant of problem $(\mathrm{P}_{\chi_1,\chi_2})$.

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